LIMITING ABSORPTION PRINCIPLE FOR THE DISSIPATIVE HELMHOLTZ EQUATION

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Abstract. Adapting Mourre's commutator method to the dissipative setting, we prove a limiting absorption principle for a class of abstract dissipative operators. A consequence is the resolvent estimates for the high frequency Helmholtz equation when trapped trajectories meet the set where the imaginary part of the potential is non-zero. We also give the resolvent estimates in Besov spaces.

1. Introduction

We consider the following Helmholtz equation:

$$\Delta A(x) + k_0^2(1 - N(x))A(x) + ik_0 a(x)A(x) = A_0 \tag{1.1}$$

This equation modelizes accurately the propagation of the electromagnetic field of a laser in material medium. Here k_0 is the wave number of the laser in the vacuum, N and a are smooth nonnegative functions representing the electronic density of material medium and the absorption coefficient of the laser energy by material medium, and A_0 is an incident known excitation (see [BLSS03]). Note that the laser wave cannot propagate in regions where $N(x) \geqslant 1$, so it is assumed that $0 \leqslant N(x) < 1$. An important application of this problem is the design of very high power laser device such as the Laser Méga-Joule in France or the National Ignition Facility in the USA.

The oscillatory behaviour of the solution makes numerical resolution very expensive. Fortunately, the wave length of the laser in the vacuum $2\pi k_0^{-1}$ is much smaller than the scale of variation of N. It is therefore relevant to consider the high frequency limit $k_0 \to \infty$. The simplified model with a constant absorption coefficient has been studied in many papers. This coefficient is either assumed to be positive (see [BCKP02, BLSS03, WZ06]) in order to be regularizing, or only nonnegative ([Wan07]), in which case the outgoing (or incoming) solution has to be chosen for A, but in both cases the non-symmetric part of the equation is in the spectral parameter, and what remains is a selfadjoint Schrödinger operator. More precisely we are led to study an equation of the form:

$$(-h^2\Delta + V(x) - (E + i\mu_h))u_h = S_h$$

where $h \sim k_0^{-1}$ is a small parameter.

When the absorption is not assumed to be constant, it has to be in the operator itself and the selfadjoint theory no longer applies. However, the anti-adjoint part is small compared to the selfadjoint Schrödinger operator, so by perturbation theory we can see that some results concerning the selfadjoint part still apply to the perturbed operator.

In this paper we study the limiting absorption principle for the following dissipative Schrödinger operator:

$$H_h = -h^2\Delta + V_1(x) - i\nu(h)V_2(x)$$

on $L^2(\mathbb{R}^n)$, where V_1 is a real function, V_2 is nonnegative and $\nu :]0,1] \to]0,1]$. Note that $\nu(h) = h$ for the Helmholtz equation.

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In the first section, we prove a uniform and dissipative version of the abstract commutator method introduced by E. Mourre in [Mou81] and developped in many papers (see for instance [PSS81, JMP84, Jen85, DJ01, GGM04] and references therein). In particular we see that the anti-adjoint part with fixed sign allows us to weaken the Mourre condition we need to prove uniform estimates and limiting absorption principle on the upper half-plane. On the contrary, the result is not valid on the other side of the real axis.

In section 2 we apply this abstract result to the dissipative Schrödinger operator in the semi-classical setting, following the idea of C. Gérard and A. Martinez ([GM88], see also [RT87] for the semi-classical limiting absorption principle). In particular we get uniform estimates of the resolvent $(H_h - z)^{-1}$ for h small enough, $\operatorname{Im} z > 0$ and $\operatorname{Re} z$ close to $E > 0$. In the selfadjoint case, the result is true if and only if E is a non-trapping energy, that is if there is no bounded classical trajectory for the hamiltonian flow associated to the symbol $p(x,\xi) = \xi^2 + V_1(x)$ of H_h. In the dissipative case, the weakened Mourre condition gives a weaker condition on the classical behaviour: we only have to assume that any bounded trajectory of energy E meets the set where $V_2 > 0$. Note that it is consistent with the selfadjoint result. Section 3 is devoted to prove that this condition is necessary (when $\nu(h) = h$, which is the case we are mainly interested in). To this purpose we use a selfadjoint dilation of the Schrödinger operator and we prove a dissipative Egorov theorem.

Finally, we show that the estimates we have proved in weighted spaces can be extended to estimates in Besov spaces, first for the abstract setting of section 2 and then for the Schrödinger operator.

2. Commutator method for a family of dissipative operators

We first recall that an operator H of domain $\mathcal{D}(H)$ in the Hilbert space $\mathcal{H}$ is said to be dissipative if:

$$\forall \varphi \in \mathcal{D}(H), \quad \operatorname{Im} \langle H\varphi, \varphi \rangle \leqslant 0$$

2.1. Uniform conjugate operators.

Let $(H_h)_{h \in]0,1]}$ be a family of dissipative operators on $\mathcal{H}$. We assume that H_h is of the form $H_h = H_0^h - iV_h$ where H_0^h is selfadjoint on a domain $\mathcal{D}_H$ independant of h and V_h is selfadjoint, nonnegative and uniformly H_0^h-bounded with relative bound less than 1:

$$\exists a \in [0,1[, \exists b \in \mathbb{R}, \forall h \in]0,1], \forall \varphi \in \mathcal{D}_H, \quad \|V_h \varphi\| \leqslant a \left\| H_0^h \varphi \right\| + b \|\varphi\| \tag{2.1}$$

For any $h \in]0,1]$ and $\varphi \in \mathcal{D}_H$, write: $\|\varphi\|_{\Gamma_h} = \left\| H_0^h \varphi \right\| + \|\varphi\|$. Consider now a family $(A_h)_{h \in]0,1]}$ of selfadjoint operators on a domain $\mathcal{D}_A$ independant of h, $J \subset \mathbb{R}$ and $(\alpha_h)_{h \in]0,1]}$ with $\alpha_h \in]0,1]$.

Definition 2.1. The family (A_h) is said to be uniformly conjugate to (H_h) on J with bounds (α_h) if the following conditions are satisfied:

(a) For every $h \in]0,1]$, $\mathcal{D}_H \cap \mathcal{D}_A$ is a core for H_0^h.

(b) e^{itA_h} maps $\mathcal{D}_H$ into itself for any $t \in \mathbb{R}$, $h \in]0,1]$, and:

$$\forall \varphi \in \mathcal{D}_H, \quad \sup_{h \in]0,1], |t| \leqslant 1} \left\| e^{itA_h} \varphi \right\|_{\Gamma_h} < \infty \tag{2.2}$$

(c) For every $h \in]0,1]$, the quadratic forms $i[H_0^h, A_h]$ and $i[V_h, A_h]$ defined on $\mathcal{D}_H \cap \mathcal{D}_A$ are bounded from below and closable. Moreover, if we denote by $i[H_0^h, A_h]^0$ and $i[V_h, A_h]^0$ their closures, then $\mathcal{D}_H \subset \mathcal{D}(i[H_0^h, A_h]^0) \cap \mathcal{D}(i[V_h, A_h]^0)$ and there exists $c \geqslant 0$ such that for $h \in]0,1]$ and $\varphi \in \mathcal{D}_H$ we have:

$$\left\| i[H_0^h, A_h]^0 \varphi \right\| + \left\| i[V_h, A_h]^0 \varphi \right\| \leqslant c\sqrt{\alpha_h}\, \|\varphi\|_{\Gamma_h} \tag{2.3}$$

(*d*) There exists $c \geqslant 0$ such that for all $\varphi, \psi \in \mathcal{D}_H \cap \mathcal{D}_A$:

$$\left| \left\langle i[H_0^h, A_h]^0 \varphi, A_h \psi \right\rangle - \left\langle A_h \varphi, i[H_0^h, A_h]^0 \psi \right\rangle \right| \leqslant c\, \alpha_h \left\| \varphi \right\|_{\Gamma_h} \left\| \psi \right\|_{\Gamma_h} \tag{2.4}$$

and similar estimates hold for the forms $[i[V_h, A_h], A_h]$ and $[V_h, A_h]$.

(*e*) There exists $C_V \geqslant 0$ such that for all $h \in]0,1]$:

$$\mathbb{1}_J(H_0^h) \left(i[H_0^h, A_h]^0 + C_V V_h \right) \mathbb{1}_J(H_0^h) \geqslant \alpha_h \mathbb{1}_J(H_0^h) \tag{2.5}$$

where $\mathbb{1}_J$ denotes the characteristic function of J and hence $(\mathbb{1}_I(H_0^h))_{I \subset \mathbb{R}}$ is the set of spectral projections for the selfadjoint operator H_0^h.

Let $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and for $J \subset \mathbb{R}$: $\mathbb{C}_{J,+} = \{z \in \mathbb{C}_+ : \operatorname{Re} z \in J\}$.

2.2. **Abstract limiting absorption principle.** We first prove a version of the quadratic estimates (see [Mou81, prop. II.5]) we need in our dissipative case:

Proposition 2.2. *Let $T = T_R - iT_I$ be a dissipative operator on $\mathcal{H}$ with T_R selfadjoint and T_I nonnegative, selfadjoint and T_R-bounded. Then for all $z \in \mathbb{C}_+$ the operator $(T - z)$ has a bounded inverse. Moreover if B is an operator such that $B^*B \leqslant T_I$ and Q is a bounded selfadjoint operator, then we have:*

$$\left\| B(T-z)^{-1} Q \right\| \leqslant \left\| Q(T-z)^{-1} Q \right\|^{\frac{1}{2}} \tag{2.6}$$

Proof. Since T_R is closed and T_I is T_R-bounded, T is closed. For $z \in \mathbb{C}_+$ and $\varphi \in \mathcal{D}(H_0)$ we have:

$$\|(T-z)\varphi\| \, \|\varphi\| \geqslant |\operatorname{Im} \langle (T-z)\varphi, \varphi \rangle| = \langle T_I \varphi, \varphi \rangle + \operatorname{Im} z \, \|\varphi\|^2 \geqslant \operatorname{Im} z \, \|\varphi\|^2$$

So $(T-z)$ is injective with closed range. We similarly prove that $\|(T^* - \overline{z})\varphi\| \geqslant \operatorname{Im} z \, \|\varphi\|$, so $\operatorname{Ran}(T-z)$ is dense in $\mathcal{H}$ and hence equal to $\mathcal{H}$, which proves that $(T-z)$ has a bounded inverse. Let $\varphi \in \mathcal{H}$. We compute:

$$\begin{aligned}
&\left\| B(T-z)^{-1} Q \varphi \right\|^2 \\
&\quad = \left\langle B^*B(T-z)^{-1} Q\varphi, (T-z)^{-1} Q\varphi \right\rangle \\
&\quad \leqslant \left\langle T_I (T-z)^{-1} Q\varphi, (T-z)^{-1} Q\varphi \right\rangle + \operatorname{Im} z \left\langle (T-z)^{-1} Q\varphi, (T-z)^{-1} Q\varphi \right\rangle \\
&\quad \leqslant \frac{1}{2i} \left\langle Q(T^* - \overline{z})^{-1} [(T^* - \overline{z}) - (T-z)](T-z)^{-1} Q\varphi, \varphi \right\rangle \\
&\quad \leqslant \left\| Q(T-z)^{-1} Q \right\| \|\varphi\|^2
\end{aligned}$$

□

Let $\langle \cdot \rangle$ denote the function $x \mapsto \sqrt{1 + |x|^2}$. We can now state and prove the main theorem of this section:

Theorem 2.3. *Let $(H_h)_{h \in]0,1]}$ be a family of dissipative operators of the form $H_h = H_0^h - iV_h$ as in section 2.1 and $(A_h)_{h \in]0,1]}$ a conjugate family to (H_h) on the open interval $J \subset \mathbb{R}$ with bounds $(\alpha_h)_{h \in]0,1]}$. Then for any closed subinterval $I \subset J$ and all $s > \frac{1}{2}$, there exists a constant $c \geqslant 0$ such that:*

$$\forall h \in]0,1], \forall z \in \mathbb{C}_{I,+}, \quad \left\| \langle A_h \rangle^{-s} (H_h - z)^{-1} \langle A_h \rangle^{-s} \right\| \leqslant \frac{c}{\alpha_h} \tag{2.7}$$

Moreover, we have for all $z, z' \in \mathbb{C}_{I,+}$:

$$\left\| \langle A_h \rangle^{-s} \left((H_h - z)^{-1} - (H_h - z')^{-1} \right) \langle A_h \rangle^{-s} \right\| \leqslant c\, \alpha_h^{-\frac{4s}{2s+1}} \left| z - z' \right|^{\frac{2s-1}{2s+1}} \tag{2.8}$$

and for $E \in J$ the limit:

$$\langle A_h \rangle^{-s} (H_h - (E + i0))^{-1} \langle A_h \rangle^{-s} = \lim_{\mu \to 0^+} \langle A_h \rangle^{-s} (H_h - (E + i\mu))^{-1} \langle A_h \rangle^{-s} \tag{2.9}$$

exists in $\mathcal{L}(\mathcal{H})$ and is a $\frac{2s-1}{2s+1}$-Hölder continuous function of E.

Remark 2.4. As in [Mou81], if we only need resolvent estimates for an operator $H = H_0 - iV$ where H_0 is selfadjoint and V is selfadjoint, nonnegative and H_0-bounded, we look for a conjugate operator which satifies the same assumptions as in definition 2.1 with a weaker Mourre condition:

$$\mathbb{1}_J(H_0)\left(i[H_0,A]^0 + C_V V\right)\mathbb{1}_J(H_0) \geqslant \alpha \mathbb{1}_J(H_0) + \mathbb{1}_J(H_0) K \mathbb{1}_J(H_0)$$

where K is a compact operator on $\mathcal{H}$. Indeed, for any $E \in J \cap \sigma_c(H_0)$ (the continuous spectrum of H_0) we can find $\delta > 0$ such that:

$$\mathbb{1}_{[E-\delta,E+\delta]}(H_0) K \mathbb{1}_{[E-\delta,E+\delta]}(H_0) \geqslant -\frac{\alpha}{2}\mathbb{1}_{[E-\delta,E+\delta]}(H_0)$$

hence A is conjugate to H on $[E-\delta, E+\delta]$ with bound $\frac{\alpha}{2}$ in the sense of definition 2.1.

The proof of theorem 2.3 follows that of the selfadjoint analog:

Proof. Let $I \subset J$ be a closed interval and $s \in \left]\frac{1}{2}, 1\right]$ (the conclusions are weaker for $s > 1$). Throughout the proof, c stands for a constant which may change but does not depend on $z \in \mathbb{C}_{I,+}$, $\varepsilon \in]0,1]$ and $h \in]0,1]$.

1. Let $\phi \in C_0^\infty(J, [0,1])$ with $\phi = 1$ in a neighborhood of I. We set $P_h = \phi(H_0^h)$ and $P_h' = (1-\phi)(H_0^h)$. We also define: $\Theta_{R,h} = i[H_0^h, A_h]^0$, $\Theta_{I,h} = i[V_h, A_h]^0$, $\Theta_h = \Theta_{R,h} - i\Theta_{I,h}$ and $\Theta_h^V = C_V V_h + \Theta_h$, C_V being given by assumption (e). Then by assumptions (c) and (2.1), Θ_h^V is H_0^h-bounded and:

$$\|\Theta_h P_h\| + \|P_h \Theta_h\| \leqslant c\sqrt{\alpha_h} \tag{2.10}$$

The operator V_h is H_0^h-bounded and $P_h \Theta_h^V P_h$ is bounded, so for all $h, \varepsilon \in]0,1]$ we can apply proposition 2.2 with $T_R = H_0^h - \varepsilon P_h \Theta_{I,h} P_h$ and $T_I = V_h + \varepsilon P_h (C_V V_h + \Theta_{R,h}) P_h$. Indeed by assumption (e) we have:

$$0 \leqslant (\sqrt{\alpha_h} P_h)^2 = \alpha_h P_h \mathbb{1}_J(H_0^h)^2 P_h \leqslant P_h (C_V V_h + \Theta_{R,h}) P_h \tag{2.11}$$

and hence T_I is nonnegative so $G_{z,h}(\varepsilon) = (H_h - i\varepsilon P_h \Theta_h^V P_h - z)^{-1}$ is well-defined for any $z \in \mathbb{C}_+$.

Then we write $Q_h(\varepsilon) = \langle A_h \rangle^{-s} \langle \varepsilon A_h \rangle^{s-1}$ and finally: $F_{z,h}(\varepsilon) = Q_h(\varepsilon) G_{z,h}(\varepsilon) Q_h(\varepsilon)$. By functional calculus we have:

$$\|Q_h(\varepsilon)\| \leqslant 1 \quad \text{and} \quad \|A_h Q_h(\varepsilon)\| = \|Q_h(\varepsilon) A_h\| = \varepsilon^{s-1} \tag{2.12}$$

and the second part of proposition 2.2 with $B = \sqrt{V_h}$ and $Q = Q_h(\varepsilon)$ for all $h, \varepsilon \in]0,1]$ gives :

$$\left\|\sqrt{V_h} G_{z,h}(\varepsilon) Q_h(\varepsilon)\right\| \leqslant \|F_{z,h}(\varepsilon)\|^{\frac{1}{2}} \tag{2.13}$$

2. By (2.11) and proposition 2.2 now applied with $B = \sqrt{\alpha_h}\sqrt{\varepsilon} P_h$, we also have:

$$\|P_h G_{z,h}(\varepsilon) Q_h(\varepsilon)\| \leqslant \frac{1}{\sqrt{\alpha_h}\sqrt{\varepsilon}} \|F_{z,h}(\varepsilon)\|^{\frac{1}{2}} \tag{2.14}$$

On the other hand:

$$\begin{aligned}
(1+\sqrt{V_h}) P_h' G_{z,h}(\varepsilon) Q_h(\varepsilon) &= (1+\sqrt{V_h}) P_h' (H_0^h - z)^{-1} (1 + i(V_h + \varepsilon P_h \Theta_h^V P_h) G_{z,h}(\varepsilon)) Q_h(\varepsilon) \\
&= (1+\sqrt{V_h}) P_h' (H_0^h - z)^{-1} Q_h(\varepsilon) \\
&\quad + i(1+\sqrt{V_h}) P_h' (H_0^h - z)^{-1} V_h G_{z,h}(\varepsilon) Q_h(\varepsilon) \\
&\quad + i\varepsilon(1+\sqrt{V_h}) P_h' (H_0^h - z)^{-1} P_h \Theta_h P_h G_{z,h}(\varepsilon) Q_h(\varepsilon) \\
&\quad + i\varepsilon C_V (1+\sqrt{V_h}) P_h' (H_0^h - z)^{-1} P_h V_h P_h G_{z,h}(\varepsilon) Q_h(\varepsilon)
\end{aligned} \tag{2.15}$$

By functional calculus and (2.1) we have :

$$\left\|(1+\sqrt{V_h}) P_h' (H_0^h - z)^{-1} (1+\sqrt{V_h})\right\| \leqslant c$$

With (2.13), (2.10) and (2.14), this proves that the first three terms of (2.15) are bounded by $c(1 + \|F_{z,h}(\varepsilon)\|^{\frac{1}{2}})$. For the last term, since $P_h\sqrt{V_h}$ is uniformly bounded, it only remains to estimate:

$$\begin{aligned}\varepsilon\left\|\sqrt{V_h}P_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| &\leqslant \varepsilon\left\|\sqrt{V_h}G_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| + \varepsilon\left\|\sqrt{V_h}P_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \\ &\leqslant \varepsilon\|F_{z,h}(\varepsilon)\|^{\frac{1}{2}} + \varepsilon\left\|(1+\sqrt{V_h})P'_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\|\end{aligned}$$

For $\varepsilon \in]0, \varepsilon_0]$, $\varepsilon_0 > 0$ small enough, we finally obtain:

$$\left\|P'_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| + \left\|\sqrt{V_h}P'_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \leqslant c\left(1 + \|F_{z,h}(\varepsilon)\|^{\frac{1}{2}}\right) \tag{2.16}$$

Together with (2.14) this gives:

$$\|F_{z,h}(\varepsilon)\| \leqslant \|G_{z,h}(\varepsilon)Q_h(\varepsilon)\| \leqslant c\left(1 + \frac{\|F_{z,h}(\varepsilon)\|^{\frac{1}{2}}}{\sqrt{\alpha_h}\sqrt{\varepsilon}}\right) \tag{2.17}$$

and hence:

$$\|F_{z,h}(\varepsilon)\| \leqslant \frac{c}{\alpha_h\varepsilon} \tag{2.18}$$

Note that by (2.1) we also have:

$$\begin{aligned}\left\|H_0^hG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| &\leqslant \frac{1}{1-a}\|H_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\| + \frac{b}{1-a}\|G_{z,h}(\varepsilon)Q_h(\varepsilon)\| \\ &\leqslant c\left(1 + \frac{\|F_{z,h}(\varepsilon)\|^{\frac{1}{2}}}{\sqrt{\alpha_h}\sqrt{\varepsilon}}\right)\end{aligned} \tag{2.19}$$

while (2.13) and (2.16) give:

$$\left\|\sqrt{V_h}PG_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \leqslant c\left(1 + \|F_{z,h}(\varepsilon)\|^{\frac{1}{2}}\right) \tag{2.20}$$

3. We now estimate the derivative of $F_{z,h}$ with report to ε:

$$\begin{aligned}\frac{d}{d\varepsilon}F_{z,h}(\varepsilon) &= iC_VQ_h(\varepsilon)G_{z,h}(\varepsilon)P_hV_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon) \\ &= iQ_h(\varepsilon)G_{z,h}(\varepsilon)P_h\Theta_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon) \\ &\quad + \frac{dQ_h(\varepsilon)}{d\varepsilon}G_{z,h}(\varepsilon)Q_h(\varepsilon) + Q_h(\varepsilon)G_{z,h}(\varepsilon)\frac{dQ_h(\varepsilon)}{d\varepsilon}\end{aligned}$$

Functional calculus gives:

$$\left\|\frac{dQ_h(\varepsilon)}{d\varepsilon}\right\| \leqslant c\varepsilon^{s-1}$$

so the last two terms can be estimated by:

$$\left\|\frac{dQ_h(\varepsilon)}{d\varepsilon}G_{z,h}(\varepsilon)Q_h(\varepsilon) + Q_h(\varepsilon)G_{z,h}(\varepsilon)\frac{dQ_h(\varepsilon)}{d\varepsilon}\right\| \leqslant c\varepsilon^{s-1}\left(1 + \frac{\|F_{z,h}(\varepsilon)\|^{\frac{1}{2}}}{\sqrt{\alpha_h}\sqrt{\varepsilon}}\right)$$

By (2.20) we have :

$$\|Q_h(\varepsilon)G_{z,h}(\varepsilon)P_hV_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon)\| \leqslant c(1 + \|F_{z,h}(\varepsilon)\|)$$

and for the second term we replace $P_h\Theta_hP_h$ by $\Theta_h - P_h\Theta_hP'_h - P'_h\Theta_hP_h - P'_h\Theta_hP'_h$, which gives:

$$iQ_h(\varepsilon)G_{z,h}(\varepsilon)P_h\Theta_hP_hG_{z,h}(\varepsilon)Q_h(\varepsilon) = D_1 + D_2 + D_3 + D_4$$

with:

$$\begin{aligned}\|D_2\| &= \left\|Q_h(\varepsilon)G_{z,h}(\varepsilon)P_h\Theta_h P_h' G_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \\ &\leqslant \|Q_h(\varepsilon)G_{z,h}(\varepsilon)\| \, \|P_h\Theta_h\| \, \left\|P_h' G_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \\ &\leqslant c\left(1+\frac{1}{\sqrt{\alpha_h}\sqrt{\varepsilon}}\|F_{z,h}(\varepsilon)\|^{\frac12}\right)\times c\sqrt{\alpha_h}\times\left(1+\|F_{z,h}(\varepsilon)\|^{\frac12}\right) \\ &\leqslant c\left(1+\frac{1}{\sqrt{\varepsilon}}\|F_{z,h}(\varepsilon)\|\right)\end{aligned}$$

D_3 is estimated similarly, while we use (2.19) for D_4:

$$\begin{aligned}\|D_4\| &= \left\|Q_h(\varepsilon)G_{z,h}(\varepsilon)P_h'\Theta_h P_h' G_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \\ &\leqslant \left\|Q_h(\varepsilon)G_{z,h}(\varepsilon)P_h'\right\| \left\|\Theta_h(H_0^h-i)^{-1}P_h'\right\| \left\|(H_0^h+i)G_{z,h}(\varepsilon)Q_h(\varepsilon)\right\| \\ &\leqslant c\left(1+\frac{1}{\sqrt{\varepsilon}}\|F_{z,h}(\varepsilon)\|\right)\end{aligned}$$

To estimate D_1, we are going to use the choice of $\Theta_{R,h}$ and $\Theta_{I,h}$ as commutators with H_h. By proposition II.6 in [Mou81], $G_{z,h}(\varepsilon)$ maps $\mathcal{D}_A$ into $\mathcal{D}_H\cap\mathcal{D}_A$, so we can compute, in the sense of quadratic forms on $\mathcal{D}_H\cap\mathcal{D}_A$:

$$\begin{aligned}&Q_h(\varepsilon)G_{z,h}(\varepsilon)\Theta_h G_{z,h}(\varepsilon)Q_h(\varepsilon)\\ &\quad= iQ_h(\varepsilon)G_{z,h}(\varepsilon)[H_h,A_h]G_{z,h}(\varepsilon)Q_h(\varepsilon)\\ &\quad= iQ_h(\varepsilon)G_{z,h}(\varepsilon)[H_h-z-i\varepsilon P_h\Theta_h^V P_h,A_h]G_{z,h}(\varepsilon)Q_h(\varepsilon)\\ &\qquad -\varepsilon Q_h(\varepsilon)G_{z,h}(\varepsilon)[P_h\Theta_h^V P_h,A_h]G_{z,h}(\varepsilon)Q_h(\varepsilon)\\ &\quad= iQ_h(\varepsilon)[A_h,G_{z,h}(\varepsilon)]Q_h(\varepsilon)\\ &\qquad -\varepsilon Q_h(\varepsilon)G_{z,h}(\varepsilon)[P_h\Theta_h^V P_h,A_h]G_{z,h}(\varepsilon)Q_h(\varepsilon)\end{aligned}\tag{2.21}$$

For $\varphi,\psi\in\mathcal{D}_H\cap\mathcal{D}_A$, we have:

$$|\langle G_{z,h}(\varepsilon)Q_h(\varepsilon)\varphi, A_hQ_h(\varepsilon)\varphi\rangle| \leqslant c\,\alpha_h^{-\frac12}\varepsilon^{s-\frac32}\|F_{z,h}(\varepsilon)\|^{\frac12}\|\varphi\|\,\|\psi\|$$

according to (2.17) and (2.12).

By proposition II.6 in [Mou81], the quadratic form $[P_h\Theta_h^V P_h, A_h]$ has the properties of $[\Theta_h^V, A_h]$ given by assumption (d). With (2.19) this proves:

$$\begin{aligned}&\varepsilon\left|\left\langle [P_h\Theta_h^V P_h,A_h]G_{z,h}(\varepsilon)Q_h(\varepsilon)\varphi, G_{z,h}(\varepsilon)^*Q_h(\varepsilon)\psi\right\rangle\right|\\ &\qquad\leqslant c\,\alpha_h\,\varepsilon\,\|G_{z,h}(\varepsilon)Q_h(\varepsilon)\varphi\|_{\Gamma_h}\|G_{z,h}(\varepsilon)Q_h(\varepsilon)\psi\|_{\Gamma_h}\\ &\qquad\leqslant c(1+\|F_{z,h}(\varepsilon)\|)\,\|\varphi\|\,\|\psi\|\end{aligned}$$

So both terms in (2.21) extend to bounded operators and:

$$\|D_1\|\leqslant c\,\alpha_h^{-\frac12}\varepsilon^{s-\frac32}\left(1+\|F_{z,h}(\varepsilon)\|^{\frac12}\right)+c\,(1+\|F_{z,h}(\varepsilon)\|)$$

and hence we have proved:

$$\left\|\frac{d}{d\varepsilon}F_{z,h}(\varepsilon)\right\|\leqslant c+\frac{c}{\sqrt{\varepsilon}}\|F_{z,h}(\varepsilon)\|+\frac{c\varepsilon^{s-\frac32}}{\sqrt{\alpha_h}}\|F_{z,h}(\varepsilon)\|^{\frac12}$$

which can also be written:

$$\left\|\frac{d}{d\varepsilon}\alpha_hF_{z,h}(\varepsilon)\right\|\leqslant c+\frac{c}{\sqrt{\varepsilon}}\|\alpha_hF_{z,h}(\varepsilon)\|+c\varepsilon^{s-\frac32}\|\alpha_hF_{z,h}(\varepsilon)\|^{\frac12}\tag{2.22}$$

4. Using lemma 3.3 in [JMP84] with (2.18) and (2.22), we get that $F_{z,h}(\varepsilon)$ can be continuously continued for $\varepsilon=0$. Furthermore, the constants in this lemma do not depend on the function but only on the estimates. Since $(\alpha_hF_{z,h}(\varepsilon))$ and its derivative with report to ε are estimated uniformly in h, we can conclude that $\alpha_hF_{z,h}(0)$ is uniformly bounded in h, which is exactly (2.7).

5. Since $F_{z,h}(\varepsilon)$ is bounded as a function of ε, (2.22) becomes:

$$\left\| \frac{d}{d\varepsilon} F_{z,h}(\varepsilon) \right\| \leqslant c\, \alpha_h^{-1} \varepsilon^{s-\frac{3}{2}}$$

and gives:

$$\| F_{z,h}(\varepsilon) - F_{z,h}(0) \| \leqslant c\, \alpha_h^{-1} \varepsilon^{s-\frac{1}{2}} \tag{2.23}$$

Moreover with (2.17) we get:

$$\left\| \frac{d}{dz} F_{z,h}(\varepsilon) \right\| \leqslant \left\| Q_h(\varepsilon) G_{z,h}(\varepsilon)^2 Q_h(\varepsilon) \right\| \leqslant \| G_{z,h}(\varepsilon) Q_h(\varepsilon) \|^2 \leqslant \frac{c}{\alpha_h^2 \varepsilon}$$

and hence for $z, z' \in \mathbb{C}_{I,+}$:

$$\left\| F_{z,h}(\varepsilon) - F_{z',h}(\varepsilon) \right\| \leqslant \frac{c\, |z - z'|}{\alpha_h^2 \varepsilon} \tag{2.24}$$

Take now $z, z' \in \mathbb{C}_{I,+}$ close enough, $h \in]0,1]$ and $\varepsilon = \alpha_h^{-\frac{2}{2s+1}} |z - z'|^{\frac{2}{2s+1}}$. Then (2.23) and (2.24) give (2.8).

In particular, for $E \in I$ the map $\mu \mapsto F_{E+i\mu,h}(0)$ has a limit for $\mu \to 0^+$, and taking the limit $\mu \to 0$ in (2.8) with $z = E + i\mu$ and $z' = E' + i\mu$ shows that the limit is Hölder-continuous with report to E and finishes the proof. □

Remark 2.5. We added the uniform estimate on $[V_h, A_h]$ in assumptions (d) because we had to put V_h in the ε-term of $G_{z,h}(\varepsilon)$ in order to use the weak Mourre estimate (2.5). But this assumption is useless if we can take $C_V = 0$ in (2.5).[1]

3. Application to the dissipative Helmholtz equation

In this section we apply the abstract Mourre theory to the dissipative Schrödinger operator. Let $V_1 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with:

$$\forall \alpha \in \mathbb{N}^n, \forall x \in \mathbb{R}^n, \quad |\partial^\alpha V_1(x)| \leqslant C_\alpha \langle x \rangle^{-\rho - |\alpha|} \tag{3.1}$$

for some $\rho > 0$ and $C_\alpha \geqslant 0$. Let $V_2 \in C^\infty(\mathbb{R}^n, \mathbb{R})$ nonnegative, with bounded derivatives (up to any order) and:

$$V_2(x) \xrightarrow[|x| \to \infty]{} 0 \tag{3.2}$$

We consider on $L^2(\mathbb{R}^n)$ the operator:

$$H_h = -h^2 \Delta + V_1 - i\nu(h) V_2$$

where $\nu(h) \in]0,1]$. We denote by $H_1^h = -h^2\Delta + V_1(x)$ the selfadjoint part of H_h, $\tilde\nu(h) = \min(1, \nu(h)/h)$ and:

$$\mathcal{O} = \{(x,\xi) \in \mathbb{R}^{2n} : V_2(x) > 0\}$$

We also write $\operatorname{Op}_h^w(a)$ for the Weyl-quantization of a symbol a (see [Rob87, Mar02, EZ]):

$$\operatorname{Op}_h^w(a) u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x - y, \xi \rangle} a\left(\frac{x+y}{2}, \xi \right) u(y)\, dy\, d\xi$$

[1] Initially, estimate $\left\| V_h (H_0^h + i)^{-1} \right\| = O(\sqrt{\alpha_h})$ was also required in assumption (c). I thank Th. Jecko for pointing out that this can be avoided.

3.1. **Hamiltonian flow.** Let $p:(x,\xi)\mapsto \xi^2+V_1(x)$ be the symbol of H_1^h, and $(x_0,\xi_0)\mapsto \phi^t(x_0,\xi_0)=(\overline{x}(t,x_0,\xi_0),\overline{\xi}(t,x_0,\xi_0))\in\mathbb{R}^{2n}$ the corresponding hamiltonian flow:

$$\begin{cases}\partial_t\overline{x}(t,x_0,\xi_0)=2\overline{\xi}(t,x_0,\xi_0)\\ \partial_t\overline{\xi}(t,x_0,\xi_0)=-\nabla V_1(\overline{x}(t,x_0,\xi_0))\\ \phi^0(x_0,\xi_0)=(x_0,\xi_0)\end{cases}$$

For $I\subset\mathbb{R}$ we introduce:

$$\Omega_b(I)=\{w\in p^{-1}(I):\{\overline{x}(t,w)\}_{t\in\mathbb{R}}\text{ is bounded}\}$$
$$\Omega_\infty^\pm(I)=\{w\in p^{-1}(I):|\overline{x}(t,w)|\xrightarrow[t\to\pm\infty]{}\infty\}$$

We recall a few basic facts about this flow:

Proposition 3.1. *(i) For $a\in C^\infty(\mathbb{R}^{2n})$ we have $\partial_t(a\circ\phi^t)=\{p,a\circ\phi^t\}$ where $\{\cdot,\cdot\}$ is the Poisson bracket.*

(ii) If $I\subset\mathbb{R}_+^*$ is closed, there exists $R_0(I)>0$ such that for any $R\geqslant R_0(I)$, a trajectory of energy in I which leaves $B_x(R)$ (in the future or in the past) cannot come back.

(iii) If $I\subset\mathbb{R}_+^*$, $p^{-1}(I)=\Omega_b(I)\cup\Omega_\infty^+(I)\cup\Omega_\infty^-(I)$.

(iv) If $I\subset\mathbb{R}_+^*$ is closed, then $\Omega_b(I)$ is compact in $\mathbb{R}^{2n}$.

(v) If $I\subset\mathbb{R}_+^*$ is open, then $\Omega_\infty^+(I)$ and $\Omega_\infty^-(I)$ are open.

3.2. **Limiting absorption principle for the dissipative Schrödinger operator.**

Theorem 3.2. *Let $E>0$ and $s>\frac12$. If all bounded trajectories of energy E meet $\mathcal{O}$, then there exists $c\geqslant 0$, $h_0>0$ and $I=[E-\delta,E+\delta]$, $\delta>0$, such that:*

(i) For all $h\in]0,h_0]$:

$$\sup_{z\in\mathbb{C}_{I,+}}\left\|\langle x\rangle^{-s}(H_h-z)^{-1}\langle x\rangle^{-s}\right\|\leqslant\frac{c}{h\tilde\nu(h)}\tag{3.3}$$

(ii) For all $h\in]0,h_0]$ and $z,z'\in\mathbb{C}_{I,+}$:

$$\left\|\langle x\rangle^{-s}\left((H_h-z)^{-1}-(H_h-z')^{-1}\right)\langle x\rangle^{-s}\right\|\leqslant c\,(h\tilde\nu(h))^{-\frac{4s}{2s+1}}\left|z-z'\right|^{\frac{2s-1}{2s+1}}\tag{3.4}$$

(iii) For $\lambda\in I$ and $h\in]0,h_0]$ the limit:

$$\langle x\rangle^{-s}(H_h-(\lambda+i0))^{-1}\langle x\rangle^{-s}=\lim_{\mu\to 0^+}\langle x\rangle^{-s}(H_h-(\lambda+i\mu))^{-1}\langle x\rangle^{-s}\tag{3.5}$$

exists in $\mathcal{L}(L^2(\mathbb{R}^n))$ and is a $\frac{2s-1}{2s+1}$-Hölder continuous function of λ.

Remark 3.3. This condition that a damping perturbation of the Schrödinger operator allows to weaken a non-trapping condition already appears in [AK07] where dispersive estimates are obtained for the Schrödinger operator on an exterior domain.

Remark 3.4. We are mainly interested in the cases $\nu(h)=h$ ($\tilde\nu(h)=1$), as mentionned in the introduction, and $\nu(h)=h^2$ ($\tilde\nu(h)=h$) which appears in the study of the high energy limit for the Schrödinger operator $-\Delta-iV_2-z$, $\operatorname{Re}z\gg 1$ (see [AK07, §1.2]).

Remark 3.5. If E is a non-trapping energy, we have the usual estimate in $O(h^{-1})$, no matter how small the anti-adjoint part is.

The proof of theorem 3.2 follows that of the selfadjoint case given in [GM88]: we find a conjugate family of operators using the quantization of an escape function and then we check that this operators can be replaced by $\langle x\rangle$ in the results of theorem 2.3. The only difference is that we need to prove a weaker Mourre estimate so we are allowed to consider a function which is not an escape function where V_2 is not zero. Let us denote:

$$A_h=\frac12(x.hD+hD.x)\tag{3.6}$$

the generator of dilations.

Proposition 3.6. *For any $r \in C_0^\infty(\mathbb{R}^{2n}, \mathbb{R})$ the operators $\tilde{\nu}(h)F_h = \tilde{\nu}(h)(A_h + \mathrm{Op}_h^w(r))$ are selfadjoint and satisfy assumptions (a) to (d) for a conjugate operator to H_h.*

The proof of this proposition is not really changed by the imaginary part of V, so we omit it. The important assumption is the Mourre estimate (e), for which we need to chose r more carefully:

Proposition 3.7. *Assume that every bounded trajectory of energy E goes through $\mathcal{O}$, then there exists $\varepsilon > 0$ and $r \in C_0^\infty(\mathbb{R}^{2n}, \mathbb{R})$ such that $\tilde{\nu}(h)F_h = \tilde{\nu}(h)(A_h + \mathrm{Op}_h^w(r))$ is conjugate to H_h on $J = [E-\varepsilon, E+\varepsilon]$ with bounds $c_0 h\tilde{\nu}(h)$, where $c_0 > 0$.*

Proof. **1.** We first remark that the assumption on bounded trajectories can be extended to a neighborhood of E: there exists $\varepsilon \in]0, E/12]$ such that any bounded trajectory of energy in $[E-3\varepsilon, E+3\varepsilon]$ meets $\mathcal{O}$. Indeed, assume that for any $n \in \mathbb{N}$ we can find w_n in the compact set $\Omega_b([E/2, 2E])$ such that $p(w_n) \to E$ and $w_n \notin \mathcal{O}^\phi$ where:

$$\mathcal{O}^\phi = \bigcup_{t\in\mathbb{R}} \phi^{-t}(\mathcal{O})$$

Maybe after extracting a subsequence we can assume that $w_n \to w \in \Omega_b([E/2, 2E])$. As p is continuous, we have $p(w) = E$, and hence $w \in \mathcal{O}^\phi$ which is open. This gives a contradiction. We set $J =]E-\varepsilon, E+\varepsilon[$, $J_2 =]E-2\varepsilon, E+2\varepsilon[$ and $J_3 =]E-3\varepsilon, E+3\varepsilon[$.

2. Let $R \geqslant R_0(\overline{J_3})$ (given in proposition 3.1) so large that $\Omega_b(\overline{J_3}) \subset B_x(R)$, where $B_x(R) = \{(x,\xi) \in \mathbb{R}^{2n} : |x| < R\}$, and:

$$|2V_1(x) + x.\nabla V_1(x)| \leqslant \frac{E}{2} \quad \text{when } |x| \geqslant R$$

Let $b \in C^\infty(\mathbb{R}^n)$ equal to $x.\xi$ outside $B_x(R+1)$ and zero in a neighborhood of $\overline{B}_x(R)$. Then, if $p(x,\xi) \in J_3$ and $|x| \geqslant R+1$ we have:

$$\{p, b\}(x,\xi) = 2p(x,\xi) - 2V_1(x) - x.\nabla V_1(x) \geqslant E \tag{3.7}$$

and $\{p, b\} = 0$ in $B_x(R)$.

3. Let $w \in \Omega_b(\overline{J_3})$ and $T_w \in \mathbb{R}$ such that $\phi^{T_w}(w) \in \mathcal{O}$. As ϕ^{T_w} is continuous, we can find $\gamma_w > 0$ and an open neighborhood $\mathcal{V}_w$ of w in $\mathbb{R}^{2n}$ such that for any $z \in \mathcal{V}_w$ we have $\phi^{T_w}(z) \in \mathcal{O}_{\gamma_w}$ where $\mathcal{O}_\gamma$ stands for $\{(x,\xi) \in \mathbb{R}^{2n} : V_2(x) > \gamma\}$. Let $\mathcal{U}_w$ be another neighborhood of w with $\overline{\mathcal{U}_w} \subset \mathcal{V}_w$, $g_w \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ be supported in $\mathcal{V}_w$ and equal to 1 on $\mathcal{U}_w$, and $f \in C^\infty(\mathbb{R}^{2n})$ defined for $z \in \mathbb{R}^{2n}$ by:

$$f_w(z) = \int_0^{T_w} g_w(\phi^{-t}(z))\, dt$$

f_w has been chosen to satisfy:

$$\begin{aligned}\{p, f_w\}(z) &= \int_0^{T_w} \{p, g_w \circ \phi^{-t}\}(z)\, dt = -\int_0^{T_w} \frac{d}{dt} g_w(\phi^{-t}(z))\, dt \\ &= g_w(z) - g_w(\phi^{-T_w}(z))\end{aligned}$$

The first term is supported in $\mathcal{V}_w$, nonnegative and equal to 1 on $\mathcal{U}_w$ while the support of the second is in $\phi^{T_w}(\mathcal{V}_w) \subset \mathcal{O}_{\gamma_w}$. In particular $\{p, f_w\}$ is compactly supported, nonnegative outside $\mathcal{O}_{\gamma_w}$ and equal to 1 in $\mathcal{U}_w \setminus \mathcal{O}_{\gamma_w}$.

As $\Omega_b(\overline{J_3})$ is compact, we can find $w_1, \dots, w_N \in \Omega_b(\overline{J_3})$ for some $N \in \mathbb{N}$ such that $\Omega_b(\overline{J_3}) \subset \mathcal{U} := \cup_{j=1}^N \mathcal{U}_{w_j}$. Let $\gamma = \min_{1\leqslant j\leqslant N} \gamma_{w_j}$ and $f = \sum_{j=1}^N f_{w_j}$. Then $\{p, f\}$ is compactly supported, nonnegative outside $\mathcal{O}_\gamma$ and greater than or equal to 1 in $\mathcal{U} \setminus \mathcal{O}_\gamma$.

4. We can find a constant $C_V \geqslant 0$ such that $\{p,f\} + C_V V_2 \geqslant 1$ on $\mathcal{O}_\gamma$, so that $\{p,f\} + C_V V_2$ is nonnegative on $\mathbb{R}^{2n}$ and at least 1 on $\mathcal{U}$.

5. Let:

$$\mathcal{U}_\pm = \Omega_\infty^\pm(J_3) \cap B_x(R+2)$$

We have:

$$\mathcal{U}_+ \cup \mathcal{U}_- \cup \mathcal{U} \cup p^{-1}(\mathbb{R} \setminus \overline{J}_2) \cup \left(\mathbb{R}^{2n} \setminus \overline{B}_x(R+1)\right) = \mathbb{R}^{2n}$$

Considering a partition of unity for this open cover of $\mathbb{R}^{2n}$ provides two functions $g_\pm \in C_0^\infty(\mathbb{R}^{2n}, [0,1])$ supported in $\mathcal{U}_\pm$ such that $g_\infty = g_+ + g_-$ is equal to 1 in a neighborhood of the compact set:

$$K_\infty = p^{-1}(\overline{J}_2) \cap \overline{B}_x(R+1) \setminus \mathcal{U}$$

There exists $T \geqslant 0$ such that for any $w \in \mathbb{R}^{2n}$ we can find a neighborhood $\mathcal{V}$ of w and $\tau_\pm \geqslant 0$ such that for any $v \in \mathcal{V}$ and $t \geqslant 0$ we have:

$$0 \leqslant g_\pm(\phi^{\pm t}(v)) \leqslant \mathbb{1}_{[\tau_\pm, T+\tau_\pm]}(t)$$

As a consequence the functions:

$$f_\pm = \mp \int_0^{+\infty} (g_\pm \circ \phi^{\pm t})\, dt$$

are well-defined, bounded (by T) and C^∞ on $\mathbb{R}^{2n}$. The same calculation as for f shows that $\{p, f_\pm\} = g_\pm \geqslant 0$. Hence we can find a constant $C_\infty \geqslant 0$ such that for $f_\infty = f_+ + f_-$ we have:

$$\{p, b + C_\infty f_\infty\} \geqslant E \quad \text{on } K_\infty \tag{3.8}$$

and we already know that $\{p, b + C_\infty f_\infty\} \geqslant \{p, b\}$ is nonnegative on $p^{-1}(\overline{J}_2) \setminus K_\infty$.

6. Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ equal to 1 on $B(R+2)$. Since we can replace ζ by $x \mapsto \zeta(\mu x)$ with μ small enough, we can assume that:

$$\|C_\infty f \{p, \zeta\}\|_{L^\infty(p^{-1}(J_2))} \leqslant 2 C_\infty T \sup_{p^{-1}(J_2)} |\xi . \nabla \zeta(x)| \leqslant \frac{E}{2}$$

With (3.7) and (3.8) this gives:

$$\{p, b + \zeta f_\infty\} \geqslant \frac{E}{2} \quad \text{on } p^{-1}(J_2) \setminus \mathcal{U} \tag{3.9}$$

and $\{p, b + \zeta f_\infty\}$ is still nonnegative on $p^{-1}(J_2)$ since $\nabla\zeta = 0$ on $\mathcal{U}$. Taking $r(x,\xi) = x.\xi - b(x,\xi) + C_\infty \zeta f_\infty + f$ then $r \in C_0^\infty(\mathbb{R}^{2n})$ and:

$$\{p, x.\xi + r\} + C_V V_2 \geqslant 2c_0 \quad \text{on } p^{-1}(J_2) \text{ with } 2c_0 = \min\left(1, \frac{E}{2}\right)$$

7. Let $F_h = A_h + \mathrm{Op}_h^w(r) = \mathrm{Op}_h^w(x.\xi + r)$. The principal symbol of the operator $ih^{-1}[H_{1,h}, F_h]$ is $\{p, x.\xi + r\}$. Let $\chi \in C_0^\infty(\mathbb{R})$ supported in J_2 and equal to 1 on J. By [Rob87] or [HR83] the operator $\chi(H_1^h)$ is a pseudo-differential operator of principal symbol $\chi \circ p$. As a consequence the principal symbol of the operator:

$$\frac{i}{h}\chi(H_1^h)[H_1^h, F_h]\chi(H_1^h) + C_V V_2 - 2c_0 \chi(H_1^h)^2$$

is nonnegative, so by Gårding inequality (see theorem 4.27 in [EZ]) there is $C \geqslant 0$ such that, after multiplication by $h\tilde{\nu}(h)$:

$$\chi(H_1^h) i \left[H_1^h, \tilde{\nu}(h) F_h\right] \chi(H_1^h) + h\tilde{\nu}(h) C_V V_2 \geqslant 2h\tilde{\nu}(h) c_0 \chi(H_1^h)^2 - Ch^2\tilde{\nu}(h)$$

Taking h small enough and multiplying by $\mathbb{1}_J(H_1^h)$ on both sides give:

$$\mathbb{1}_J(H_1^h)\left(i\left[H_1^h, \tilde{\nu}(h) F_h\right] + h\tilde{\nu}(h) C_V V_2\right)\mathbb{1}_J(H_1^h) \geqslant h\tilde{\nu}(h) c_0 \mathbb{1}_J(H_1^h)^2$$

Then $(\nu(h) - h\tilde{\nu}(h))\mathbb{1}_J(H_1^h)C_V V_2 \mathbb{1}_J(H_1^h) \geqslant 0$ so we have:

$$\mathbb{1}_J(H_1^h)\left(i\left[H_1^h, \tilde{\nu}(h)F_h\right] + C_V\nu(h)V_2\right)\mathbb{1}_J(H_1^h) \geqslant h\tilde{\nu}(h)c_0\mathbb{1}_J(H_1^h)^2$$

which is the Mourre estimate we need. Note that if E is non-trapping we can take $f = 0$, $C_V = 0$, and use the estimate:

$$\mathbb{1}_J(H_1^h)i[H_1^h, F_h]\mathbb{1}_J(H_1^h) \geqslant hc_0\mathbb{1}_J(H_1^h)^2$$

even if $h \geqslant \nu(h)$, which justifies remark 3.5. □

This proposition shows that for any closed subinterval I of J theorem 3.2 is true with $\langle\tilde{\nu}(h)F_h\rangle^{-s}$ instead of $\langle x\rangle^{-s}$. The operator $\langle\tilde{\nu}(h)F_h\rangle^{s}\langle\tilde{\nu}(h)A_h\rangle^{-s}$ is bounded uniformly in h (this is true for $s = 0$ and $s = 1$ hence for any $s \in [0,1]$ by complex interpolation), so conclusions of theorem 3.2 are valid with $\langle\tilde{\nu}(h)A_h\rangle^{-s}$. Now write:

$$(H_h - z)^{-1} = (H_h - i)^{-1} - (z-i)(H_h - i)^{-2} + (z-i)^2(H_h - i)^{-1}(H_h - z)^{-1}(H_h - i)^{-1}$$

Since $\langle\tilde{\nu}(h)A_h\rangle^{s}(H_h - i)^{-1}\langle x\rangle^{-s}$ is uniformly bounded (see [PSS81, lemma 8.2]), this gives:

$$\begin{aligned}
&\left\|\langle x\rangle^{-s}(H_h - z)^{-1}\langle x\rangle^{-s}\right\| \\
&\quad\leqslant c + c\left\|\langle x\rangle^{-s}(H_h - i)^{-1}(H_h - z)^{-1}(H_h - i)^{-1}\langle x\rangle^{-s}\right\| \qquad (3.10)\\
&\quad\leqslant c + c\left\|\langle x\rangle^{-s}(H_h - i)^{-1}\langle\tilde{\nu}(h)A_h\rangle^{s}\right\|\left\|\langle\tilde{\nu}(h)A_h\rangle^{-s}(H_h - z)^{-1}\langle\tilde{\nu}(h)A_h\rangle^{-s}\right\| \\
&\qquad\qquad \times\left\|\langle\tilde{\nu}(h)A_h\rangle^{s}(H_h - i)^{-1}\langle x\rangle^{-s}\right\| \\
&\quad\leqslant \frac{c}{h\tilde{\nu}(h)}
\end{aligned}$$

where c does not depend on $z \in \mathbb{C}_{I,+}$ with $\operatorname{Im} z \leqslant 1$. This is (3.3). Then (3.4) and hence existence of the limit (3.5) follow similarly.

4. Necessity of the condition on trapped trajectories

We consider in this section the operator $H_h = -h^2\Delta + V_1 - ihV_2$ we introduced to study the Helmholtz equation. We prove that our assumption that every bounded trajectory of energy E should meet the open set $\mathcal{O}$ is actually necessary in order to have the uniform estimates and the limiting absorption principle as in theorem 3.2. When $V_2 = 0$, this is proved in [Wan87].

Theorem 4.1. *Assume that for some* $s \in \left]\frac{1}{2}, \frac{1+\rho}{2}\right[$ *(*$\rho > 0$ *given by* (3.1)*), there exists* $\varepsilon, h_0 > 0$ *such that the limit:*

$$\langle x\rangle^{-s}(H_h - (\lambda + i0))^{-1}\langle x\rangle^{-s}$$

exists for all $\lambda \in J =]E - \varepsilon, E + \varepsilon[$ *and* $h \in]0, h_0]$ *with the estimates:*

$$\left\|\langle x\rangle^{-s}(H_h - z)^{-1}\langle x\rangle^{-s}\right\| \leqslant \frac{c}{h}$$

uniformly in $z \in \mathbb{C}_{J,+}$ *and* $h \in]0, h_0]$*, then every bounded trajectory of energy* E *goes through* $\mathcal{O}$*.*

To prove this theorem we use the contraction semigroup generated by H_h (given by Hille-Yosida theorem, see for instance theorem 3.5 in [EN06]):

$$U_h(t) = e^{-\frac{it}{h}H_h}, \quad t \geqslant 0$$

We first need a dissipative version of the Egorov theorem. Let $q \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^{2n})$ be defined by:

$$q(t,w) = \exp\left(-2\int_0^t V_2(\phi^s(w))\,ds\right)$$

(where $V_2(x,\xi)$ means $V_2(x)$ for $(x,\xi) \in \mathbb{R}^{2n}$).

Theorem 4.2. *Let $a \in C^\infty(\mathbb{R}^{2n})$ be a symbol whose derivatives are bounded (in $L^\infty(\mathbb{R}^n)$). Then for all $t \geqslant 0$ we have:*

$$U_h(t)^* \operatorname{Op}_h^w(a) U_h(t) = \operatorname{Op}_h^w\big((a \circ \phi^t) q(t)\big) + hR(t,h) \tag{4.1}$$

where R is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ uniformly in $h \in]0,1]$ and t in a compact subset of $\mathbb{R}_+$.

Remark 4.3. More precisely, we prove that there exists a family $(b(\tau,h))_{\tau\geqslant 0}$ of classical symbols with bounded derivatives (uniformly for τ in a compact subset of $\mathbb{R}_+$) such that for all $t \geqslant 0$:

$$R(t,h) = \int_0^t U_h(\tau)^* \operatorname{Op}_h^w(b(\tau,h)) U_h(\tau)\, d\tau \tag{4.2}$$

Remark 4.4. If we replace one of the $U_h(t)$ by $U_1^h(t) = e^{-\frac{it}{h}H_1^h}$ in the left-hand side of (4.1) then we have to replace q by

$$q_1 : (x,\xi) \mapsto \exp\left(-\int_0^t V_2(\phi^s(w))\, ds\right) \tag{4.3}$$

in the right-hand side (with the two occurences of $U_h(t)$ replaced by $U_1^h(t)$ and q replaced by 1, theorem 4.2 is just the usual Egorov theorem).

Proof. We follow the proof of the usual Egorov theorem (see for instance [Rob87, § IV.4]). Let $t \geqslant 0$. For $\tau \in [0,t]$ and $w \in \mathbb{R}^{2n}$ write:

$$\tilde a(\tau, w) = a(\phi^{t-\tau}(w)) \exp(S(\tau,w)) \quad \text{where} \quad S(\tau,w) = -2\int_\tau^t V_2(\phi^{s-\tau}(w))\, ds$$

and:

$$B_h(\tau) = U_h(\tau)^* \operatorname{Op}_h^w(\tilde a(\tau)) U_h(\tau)$$

so that the estimate we have to prove is: $B_h(t) - B_h(0) = O(h)$ in $\mathcal{L}(L^2(\mathbb{R}^n))$. We have:

$$\begin{aligned}
\partial_\tau \tilde a(\tau) &= -\{p, a\circ\phi^{t-\tau}\}\exp(S(\tau)) + 2\left(V_2 + \int_\tau^t \{p, V_2\circ\phi^{s-\tau}\}\, ds\right)\tilde a(\tau)\\
&= -\{p, a\circ\phi^{t-\tau}\}\exp(S(\tau)) + 2V_2\tilde a(\tau) - \{p, S(\tau)\}\,\tilde a(\tau)\\
&= 2V_2 \tilde a(\tau) - \{p, \tilde a(\tau)\}
\end{aligned}$$

The function $\tau \mapsto B_h(\tau)$ is of class C^1 in the weak sense and:

$$B_h'(\tau) = U_h(\tau)^* \tilde B_h(\tau) U_h(\tau)$$

with:

$$\begin{aligned}
\tilde B_h(\tau) &= \frac{i}{h}[H_1^h, \operatorname{Op}_h^w(\tilde a(\tau))] - V_2 \operatorname{Op}_h^w(\tilde a(\tau)) - \operatorname{Op}_h^w(\tilde a(\tau)) V_2 + \operatorname{Op}_h^w(\partial_\tau \tilde a(\tau))\\
&= \operatorname{Op}_h^w(c(\tau,h))
\end{aligned}$$

for some classical symbol $c(\tau,h) = \sum_{j\in\mathbb{N}} h^j c_j(\tau)$, and in particular:

$$c_0(\tau) = \{p, \tilde a(\tau)\} - V_2\tilde a(\tau) - \tilde a(\tau) V_2 + \partial_\tau \tilde a(\tau) = 0$$

Setting $b = h^{-1}c$ we get (4.1)-(4.2), in the weak sense and hence in $\mathcal{L}(L^2(\mathbb{R}^n))$. □

Proposition 4.5. *Assume that the assumptions of theorem 4.1 are satisfied. Then for any $\chi \in C_0^\infty(\mathbb{R})$ supported in J there exists $c \geqslant 0$ such that for all $h \in]0,h_0]$ and $z \in \mathbb{C}_+$ we have:*

$$\left\| \langle x\rangle^{-s} \chi(H_1^h)(H_h - z)^{-1}\chi(H_1^h)\langle x\rangle^{-s}\right\| \leqslant \frac{c}{h} \tag{4.4}$$

Remark 4.6. We have similar estimates for $(H_h^* - \overline{z})^{-1}$.

Proof. First, we can find $c \geqslant 0$ such that estimate (4.4) holds for $z \in \mathbb{C}_{J,+}$ by assumption and uniform boundedness of $\langle x\rangle^{\mp s}\chi(H_1^h)\langle x\rangle^{\pm s}$ with report to h (note that this last statement holds for $s=0$ by functional calculus and for $s=1$, we use the fact that $\chi(H_1^h)$ is a pseudo-differential operator whose symbol has bounded derivatives and $[x, \mathrm{Op}_h^w(b)] = -ih\,\mathrm{Op}_h^w(\partial_\xi b)$; then the claim follows for any $s \in [0,1]$ by complex interpolation).

Then, there exists $\delta > 0$ such that for all $z \in \mathbb{C}_{\mathbb{R}\setminus J,+}$ we have $d(z, \operatorname{supp}\chi) \geqslant \delta$. As a consequence, the operator $\chi(H_1^h)(H_1^h - z)^{-1}$ is bounded uniformly in $z \in \mathbb{C}_{\mathbb{R}\setminus J,+}$ and $h \in]0, h_0]$. Hence, using twice the resolvent equation, we can write:

$$\begin{aligned}
&\left\|\chi(H_1^h)(H_h - z)^{-1}\chi(H_1^h)\right\| \\
&\quad\leqslant c + h^2\left\|\chi(H_1^h)(H_1^h - z)^{-1}V_2(H_h - z)^{-1}V_2(H_1^h - z)^{-1}\chi(H_1^h)\right\| \\
&\quad\leqslant c\left(1 + h\left\|\sqrt{hV_2}(H_h - z)^{-1}\sqrt{hV_2}\right\|\right) \\
&\quad\leqslant c
\end{aligned}$$

where the last step is given by proposition 2.2 applied with $T = H_h = H_1^h - ihV_2$ and $B = Q = \sqrt{hV_2}$. □

Proposition 4.7. *Assume that the assumptions of theoreme 4.1 are satisfied. Then for any $\chi \in C_0^\infty(\mathbb{R})$ supported in J there exists $C_\chi \geqslant 0$ such that for all $\psi \in L^2(\mathbb{R}^n)$ and $h \in]0, h_0]$ we have:*

$$\int_0^{+\infty}\left\|\langle x\rangle^{-s}\chi(H_1^h)U_h(t)\psi\right\|^2 dt \leqslant C_\chi\|\psi\|^2 \tag{4.5}$$

Proof. Let K_h be the selfadjoint dilation of H_h on the Hilbert space $\mathcal{K} \supset L^2(\mathbb{R}^n)$ given in appendix A. Let P be the orthogonal projection of $\mathcal{K}$ on $L^2(\mathbb{R}^n)$ and $A_h = \langle x\rangle^{-s}\chi(H_1^h) \in \mathcal{L}(\mathcal{K})$, where operators on $L^2(\mathbb{R}^n)$ are extended by 0 on $L^2(\mathbb{R}^n)^\perp \subset \mathcal{K}$. Let $\varphi = (\varphi_0, \varphi_\perp) \in \mathcal{K} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)^\perp$. For $z \in \mathbb{C}_+$ we have:

$$\begin{aligned}
&\left|\left\langle A_h^*\varphi, \left((K_h - z)^{-1} - (K_h - \overline{z})^{-1}\right)A_h^*\varphi\right\rangle_{\mathcal{K}}\right| \\
&\quad= \left|\left\langle \varphi_0, \langle x\rangle^{-s}\chi(H_1^h)\left((H_h - z)^{-1} - (H_h^* - \overline{z})^{-1}\right)\chi(H_1^h)\langle x\rangle^{-s}\varphi_0\right\rangle_{L^2(\mathbb{R}^n)}\right| \\
&\quad\leqslant \frac{2c}{h}\|\varphi_0\|^2_{L^2(\mathbb{R}^n)} \leqslant \frac{2c}{h}\|\varphi\|^2_{\mathcal{K}}
\end{aligned}$$

where c is given by proposition 4.5. The same applies if $\operatorname{Im} z < 0$, so by theorem XIII.25 in [RS79], where h-dependance has to be checked for our semiclassical setting, this proves that A_h is K_h-smooth and:

$$\sup_{h\in]0,h_0]}\sup_{\|\varphi\|=1}\int_{\mathbb{R}}\left\|A_h e^{-\frac{it}{h}K_h}\varphi\right\|^2_{\mathcal{L}(\mathcal{K})} dt < \infty \tag{4.6}$$

But for $\psi \in L^2(\mathbb{R}^n)$ (which we identify with $(\psi, 0) \in \mathcal{K}$), $h \in]0, h_0]$ and $t \geqslant 0$ we have:

$$\left\|\langle x\rangle^{-s}\chi(H_1^h)U_h(t)\psi\right\|_{\mathcal{L}(L^2(\mathbb{R}^n))} = \left\|\langle x\rangle^{-s}\chi(H_1^h)Pe^{-\frac{it}{h}K_h}P\psi\right\|_{\mathcal{L}(\mathcal{K})} = \left\|A_h e^{-\frac{it}{h}K_h}\psi\right\|_{\mathcal{L}(\mathcal{K})}$$

so (4.6) gives (4.5). □

Proposition 4.8. *Let $T \geqslant 0$ and $\chi \in C_0^\infty$ as in proposition 4.5. There exists $h_T > 0$ and $C'_\chi \geqslant 0$ such that for any $\psi \in L^2(\mathbb{R}^n)$ and $h \in]0, h_T]$ we have:*

$$\int_0^T\left\|\langle x\rangle^{-s}\chi(H_1^h)U_1^h(t)Q_h(T)\psi\right\|^2 dt \leqslant C'_\chi\|\psi\|^2 \tag{4.7}$$

where $Q_h(T) = \mathrm{Op}_h^w(q_1(T))$, q_1 being defined in (4.3).

Proof. According to Egorov theorem applied with the symbol $a(x,\xi)=1$ we have:

$$U_1^h(-t)U_h(t) = Q_h(t) + hR(t,h)$$

where R is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ uniformly for $h\in]0,1]$ and $t\in[0,T]$. On the other hand, writing $Q_h(t,T) = \mathrm{Op}_h^w(q_1(t,T))$ with $q_1(t,T) = \left(e^{-\int_t^T V_2\circ\phi^\tau\,d\tau}\right)$ for $t\in[0,T]$ we have by theorem 5.1 in [EZ]:

$$\|Q_h(t,T)\| \leqslant C + \underset{h\to 0}{O}(\sqrt{h}) \quad \text{and} \quad Q_h(T) = Q_h(t,T)Q_h(t) + \underset{h\to 0}{O}(h)$$

where C does not depend on t,T and h, and the sizes of the remainders in $\mathcal{L}(L^2(\mathbb{R}^n))$ depend on T but can be estimated uniformly on $t\in[0,T]$. Then if $\|\psi\|=1$ we have:

$$\begin{aligned}
&\int_0^T \left\| \langle x\rangle^{-s} \chi(H_1^h)U_1^h(t)Q_h(T)\psi\right\|^2 dt\\
&\leqslant \int_0^T \left\| \langle x\rangle^{-s} \chi(H_1^h)U_1^h(t)Q_h(t,T)Q_h(t)\psi\right\|^2 dt + \underset{h\to 0}{O}(h)\\
&\leqslant \int_0^T \left\| \langle x\rangle^{-s} \chi(H_1^h)Q(2t,T+t)U_1^h(t)Q_h(t)\psi\right\|^2 dt + \underset{h\to 0}{O}(h)\\
&\leqslant \int_0^T \left\| Q(2t,T+t)\langle x\rangle^{-s} \chi(H_1^h)U_h(t)\psi\right\|^2 dt + \underset{h\to 0}{O}(h)\\
&\leqslant \left(C + \underset{h\to 0}{O}(\sqrt{h})\right)\int_0^T \left\| \langle x\rangle^{-s} \chi(H_1^h)U_h(t)\psi\right\|^2 dt + \underset{h\to 0}{O}(h)\\
&\leqslant CC_\chi + \underset{h\to 0}{O}(\sqrt{h})
\end{aligned}$$

where C_χ is given by proposition 4.7. The remainder is uniformly bounded in ψ so we can chose $h_T>0$ small enough to make it less than 1 and the result follows with $C'_\chi = CC_\chi + 1$. □

We can now prove theorem 4.1 as in [Wan87]:

Proof of theorem 4.1. Let A_h be the generator of dilations defined in (3.6) and $\chi,\varphi,\psi\in C_0^\infty(\mathbb{R})$ supported in J such that $\chi(E)=1$ and $\chi(\lambda)=\lambda\varphi(\lambda)\psi(\lambda)$ for all $\lambda\in\mathbb{R}$. We have:

$$H_1^hU_1^h(T) = \frac{1}{2T}\left([A_h,U_1^h(T)] + \int_0^T U_1^h(T-t)W(x)U_1^h(t)\,dt\right)$$

where $W(x) = -2V_1(x) - x.\nabla V_1(x)$ and hence there exists $c\geqslant 0$ such that for all $T\geqslant 0$ and $h\in]0,h_T]$ ($h_T>0$ depends on T):

$$\begin{aligned}
&\left\| \langle x\rangle^{-s} Q_h(T)\chi(H_1^h)U_1^h(T)Q_h(T)\langle x\rangle^{-s}\right\| \\
&\qquad = \left\| \langle x\rangle^{-s} Q_h(T)\varphi(H_1^h)H_1^hU_1^h(T)\psi(H_1^h)Q_h(T)\langle x\rangle^{-s}\right\|\\
&\qquad \leqslant \frac{c}{T}(1+\|F_h(T)\|)
\end{aligned} \tag{4.8}$$

where:

$$F_h(T) = \int_0^T \langle x\rangle^{-s} Q_h(T)\varphi(H_1^h)U_1^h(T-t)W(x)U_1^h(t)\psi(H_1^h)Q_h(T)\langle x\rangle^{-s}\,dt$$

Indeed, we have $\|Q_h(T)\|\leqslant C + O(\sqrt{h})$, hence for $h\in]0,h_T]$ with $h_T>0$ small enough we have $\|Q_h(T)\|\leqslant 2C$. Furthermore A_h is uniformly H_1^h-bounded, so we have:

$$\left\| \langle x\rangle^{-s} Q_h(T)\varphi(H_1^h)[A_h,U_1^h(T)]\psi(H_1^h)Q_h(T)\langle x\rangle^{-s}\right\| \leqslant c$$

uniformly in $T\geqslant 0$ and $h\in]0,h_T]$.

Let us now chose $\theta \in C_0^\infty(\mathbb{R}^n)$ with support in $B(0,2)$ and equal to 1 on $B(0,1)$, and define $W_1(T,x) = W(x)\theta(x/T)$, $W_2(T,x) = W(x) - W_1(T,x)$ and $F_j^h(T)$ with the same expression as $F_h(T)$ with W replaced by W_j ($j = 1,2$). As W decays like V_1 (see (3.1)), there exists $c \geqslant 0$ such that for all $T \geqslant 0$ and $h \in]0, h_T]$ we have $\left\| F_2^h(T) \right\| \leqslant cT^{1-\rho}$. To estimate F_1^h we compute, for $\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} = 1$:

$$\begin{aligned}\left| \left\langle F_1^h(T) f, g \right\rangle \right| &\leqslant \int_0^T \left\| \langle x\rangle^{-s} U_1^h(t)\psi(H_1^h) Q_h(T) \langle x\rangle^{-s} f \right\| \left\| \langle x\rangle^{2s} W_1(t,x) \right\| \\ &\qquad \times \left\| \langle x\rangle^{-s} U_1^h(T-t)\varphi(H_1^h) Q_h(T) \langle x\rangle^{-s} g \right\| \, dt \\ &\leqslant cT^{2s-\rho} \int_0^T \left\| \langle x\rangle^{-s} \psi(H_1^h) U_1^h(t) Q_h(T) \langle x\rangle^{-s} f \right\|^2 dt \\ &\qquad \times \int_0^T \left\| \langle x\rangle^{-s} \varphi(H_1^h) U_1^h(T-t) Q_h(T) \langle x\rangle^{-s} g \right\|^2 dt \\ &\leqslant c\, T^{2s-\rho}\end{aligned}$$

where c is independant of $T \geqslant 0$ and $h \in]0, h_T]$. Finally we have:

$$\|F_h(T)\| \leqslant cT^{1-\delta} \tag{4.9}$$

with $\delta = \min(1 + \rho - 2s, \rho) > 0$ and $c \geqslant 0$ independant of $T \geqslant 0$ and $h \in]0, h_T]$.

Let $(z,\zeta) \in \Omega_b(E)$ (if $\Omega_b(E)$ is empty then there is nothing to prove) and $T \geqslant 0$. Let $W_h(z,\zeta) = \exp\left(ih^{-\frac{1}{2}}(\zeta.x - z.D) \right)$ (see [Wan85, § 3.1]) and:

$$G_h(T) = W_h(z,\zeta)^* \left\langle h^{\frac{1}{2}} x \right\rangle^{-s} R_h(T) \chi(P_1^h) V_h(T) R_h(T) \left\langle h^{\frac{1}{2}} x \right\rangle^{-s} V_h(-T) W_h(z,\zeta)$$

where $P_1^h = -h\Delta + V_1(h^{\frac{1}{2}} x)$, $V_h(T) = \exp\left(-\frac{iT}{h} P_1^h\right)$ and $R_h(T) = q_1(T)^w(h^{\frac{1}{2}} x, h^{\frac{1}{2}} \xi)$. These three operators are conjugate to H_1^h, $U_h(T)$ and $Q_h(T)$ by the unitary transformation $f \mapsto \left(x \mapsto h^{\frac{n}{4}} f(h^{\frac{1}{2}} x) \right)$, so for $T \geqslant 0$ and $h \in]0, h_T]$ we have by (4.8) and (4.9):

$$\begin{aligned}\|G_h(T)\| &= \left\| \left\langle h^{\frac{1}{2}} x \right\rangle^{-s} R_h(T) \chi(P_1^h) V_h(T) R_h(T) \left\langle h^{\frac{1}{2}} x \right\rangle^{-s} \right\| \\ &= \left\| \langle x\rangle^{-s} Q_h(T) \chi(H_1^h) U_1^h(T) Q_h(T) \langle x\rangle^{-s} \right\| \\ &\leqslant cT^{-\delta}\end{aligned}$$

where c does not depend on T and $h \in]0, h_T]$. On the other hand, using [Wan85, lemma 3.1] and [Wan86, theorem 4.2] we have:

$$\begin{aligned}G(T) &= \left\langle h^{\frac{1}{2}} x + z \right\rangle^{-s} q_1(T)^w(h^{\frac{1}{2}} x + z, h^{\frac{1}{2}} D + \zeta)\, (\chi \circ p)^w(h^{\frac{1}{2}} x + z, h^{\frac{1}{2}} D + \zeta) \\ &\qquad \times W_h(z,\zeta)^* V_h(T) q_1(T)^w(h^{\frac{1}{2}} x, h^{\frac{1}{2}} D) \left\langle h^{\frac{1}{2}} x \right\rangle^{-s} V_h(-T) W_h(z,\zeta) + \underset{h\to 0}{O}(h) \\ &\xrightarrow[h\to 0]{} \langle z\rangle^{-s} q_1(T,z,\zeta) \chi(p(z,\zeta)) q_1(T, \phi^T(z,\zeta)) \left\langle \overline{x}(T,z,\zeta) \right\rangle^{-s}\end{aligned}$$

This proves:

$$\langle z\rangle^{-s} q_1(T,z,\zeta) q_1(T, \phi^T(z,\zeta)) \left\langle \overline{x}(T,z,\zeta) \right\rangle^{-s} \leqslant cT^{-\delta}$$

where c does not depend on T, but $\overline{x}(T,z,\zeta)$ stays in a bounded subset of $\mathbb{R}^n$, so we must have:

$$q_1(T,z,\zeta) q_1(T, \phi^T(z,\zeta)) \xrightarrow[T\to+\infty]{} 0$$

which, by definition of q_1, cannot be true unless the classical trajectory starting from (z,ζ) goes through $\mathcal{O}$ (see (4.3)). $\square$

5. Uniform resolvent estimates in Besov spaces

In order to obtain in Besov spaces the resolvent estimates we proved in weighted spaces, we need another resolvent estimate (see proposition 5.2). We begin with a lemma which turns properties on $G_{z,h}(\varepsilon) = (H_h - i\varepsilon P_h \Theta_h^V P_h - z)^{-1}$ (see section 2) into properties on $(H_h - i\varepsilon\Theta_h^V - z)^{-1}$:

Lemma 5.1. *With assumptions and notations of theorem 2.3, for all $h, \varepsilon \in]0,1]$ and $z \in \mathbb{C}_{J,+}$ the operator $(H_h - i\varepsilon\Theta_h^V - z)$ has a bounded inverse (denoted by $G^1_{z,h}(\varepsilon)$) which satisfies the following estimates:*

$$\left\|G^1_{z,h}(\varepsilon)\right\| + \left\|H_0^h G^1_{z,h}(\varepsilon)\right\| \leqslant \frac{c}{\alpha_h \varepsilon} \tag{5.1}$$

$$\left\|G^1_{z,h}(\varepsilon)\left\langle A_h\right\rangle^{-1}\right\| + \left\|H_0^h G^1_{z,h}(\varepsilon)\left\langle A_h\right\rangle^{-1}\right\| \leqslant \frac{c}{\alpha_h\sqrt{\varepsilon}} \tag{5.2}$$

$$\left\|\sqrt{V_h}G^1_{z,h}(\varepsilon)\right\| \leqslant \frac{c}{\sqrt{\alpha_h}\sqrt{\varepsilon}} \tag{5.3}$$

$$\left\|\sqrt{V_h}G^1_{z,h}(\varepsilon)\left\langle A_h\right\rangle^{-1}\right\| \leqslant c \tag{5.4}$$

where c is independant of $\varepsilon, h \in]0,1]$ and $z \in \mathbb{C}_{I,+}$ for some closed subinterval I of J.

Proof. We keep all the notations of the proof of theorem 2.3, in particular $P_h = \phi(H_0^h)$, $P_h' = 1 - P_h$, $G_{z,h}(\varepsilon) = (H_h - i\varepsilon P_h\Theta_h^V P_h)^{-1}, \dots$ Applying proposition 2.2 with $B = \sqrt{\alpha_h}\sqrt{\varepsilon}P_h$ and $Q = P_h$ gives:

$$\left\|P_h G_{z,h}(\varepsilon)P_h\right\| \leqslant \frac{1}{\alpha_h\varepsilon}$$

Calculations (2.15)-(2.16) with $Q_h(\varepsilon)$ replaced by P_h and P_h' show:

$$\left\|P_h' G_{z,h}(\varepsilon)P_h\right\| \leqslant \frac{c}{\sqrt{\alpha_h}\sqrt{\varepsilon}}, \quad \left\|P_h' G_{z,h}(\varepsilon)P_h'\right\| \leqslant c \tag{5.5}$$

We also have $\left\|P_h G_{z,h}(\varepsilon)P_h'\right\| \leqslant \frac{c}{\sqrt{\alpha_h}\sqrt{\varepsilon}}$ and hence:

$$\left\|G_{z,h}(\varepsilon)\right\| + \left\|H_0^h G_{z,h}(\varepsilon)\right\| \leqslant \frac{c}{\alpha_h\varepsilon} \tag{5.6}$$

Now three applications of proposition 2.2 with $B = \sqrt{V_h}$ give:

$$\left\|\sqrt{V_h}G_{z,h}(\varepsilon)\left\langle A_h\right\rangle^{-1}\right\| + \left\|\sqrt{V_h}G_{z,h}(\varepsilon)P_h'\right\| \leqslant c, \quad \left\|\sqrt{V_h}G_{z,h}(\varepsilon)\right\| \leqslant \frac{c}{\sqrt{\alpha_h}\sqrt{\varepsilon}} \tag{5.7}$$

Then, as in [JMP84], we prove that:

$$G'_{z,h}(\varepsilon) = G_{z,h}(\varepsilon) + i\varepsilon G_{z,h}(\varepsilon)P_h'(1 - i\varepsilon\Theta_h^V P_h G_{z,h}(\varepsilon)P_h')^{-1}\Theta_h^V P_h G_{z,h}(\varepsilon)$$

is well-defined for ε small enough and is a bounded inverse of $(H_h - i\varepsilon\Theta_h^V P_h)$ which satisfies estimates (5.5)-(5.7) as $G_{z,h}(\varepsilon)$. Then it remains to define:

$$G^1_{z,h}(\varepsilon) = G'_{z,h}(\varepsilon) + i\varepsilon G'_{z,h}(\varepsilon)\Theta_h^V(1 - i\varepsilon P_h' G'_{z,h}(\varepsilon)\Theta_h^V)^{-1}P_h' G'_{z,h}(\varepsilon)$$

for ε small enough and check the conclusions of the lemma. □

Proposition 5.2. *Let $s > 1$ and I a closed subinterval of J. Then there exists $c \geqslant 0$ such that for all $z \in \mathbb{C}_{I,+}$ and $h \in]0,1]$:*

$$\left\|\mathbb{1}_{\mathbb{R}_-}(A_h)(H_h - z)^{-1}\left\langle A_h\right\rangle^{-s}\right\| \leqslant \frac{c}{\alpha_h} \tag{5.8}$$

Proof. We follow the proof of theorem 2.3 in [JMP84]. Let

$$\tilde{F}_{z,h}(\varepsilon) = \mathbb{1}_{\mathbb{R}_-}(A_h)\exp(\varepsilon A_h)G^1_{z,h}(\varepsilon)\left\langle A_h\right\rangle^{-s}$$

By (5.2) we already know that:

$$\left\| \tilde{F}_{z,h}(\varepsilon) \right\| \leqslant \frac{c}{\alpha_h \sqrt{\varepsilon}} \tag{5.9}$$

Then we compute in the sense of quadratic forms on $\mathcal{D}_H \cap \mathcal{D}_A$:

$$\begin{aligned}
\frac{d}{d\varepsilon} \tilde{F}_{z,h}(\varepsilon) &= \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} A_h G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \\
&\quad + i \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) (C_V V_h + i[H_h, A_h]) G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \\
&= \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) A_h \langle A_h \rangle^{-s} \\
&\quad + i C_V \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) V_h G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \\
&\quad - i\varepsilon \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) [\Theta_h^V, A] G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s}
\end{aligned}$$

We use complex interpolation to estimate the first term:

$$\begin{aligned}
&\left\| \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) \langle A_h \rangle^{1-s} \right\| \\
&\qquad \leqslant \left\| \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\|^{1-\frac{1}{s}} \left\| \mathbb{1}_{\mathbb{R}_-}(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) \right\|^{\frac{1}{s}} \\
&\qquad \leqslant c \, \alpha_h^{-\frac{1}{s}} \varepsilon^{-\frac{1}{s}} \left\| \tilde{F}_{z,h}(\varepsilon) \right\|^{1-\frac{1}{s}}
\end{aligned}$$

For the second term we write:

$$\begin{aligned}
\left\| \chi_-(A_h) e^{\varepsilon A_h} G^1_{z,h}(\varepsilon) V_h G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\| &\leqslant \left\| G^1_{z,h}(\varepsilon) \sqrt{V_h} \right\| \left\| \sqrt{V_h} G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\| \\
&\leqslant \frac{c}{\alpha_h \sqrt{\varepsilon}}
\end{aligned}$$

and finally, by assumption (d) and (5.1)-(5.2):

$$\begin{aligned}
\varepsilon \left\| G^1_{z,h}(\varepsilon) [\Theta_h^V, A] G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\| &\leqslant c \, \varepsilon \alpha_h \left\| G^1_{z,h}(\varepsilon) \right\|_{\Gamma_h} \left\| G^1_{z,h}(\varepsilon) \langle A_h \rangle^{-s} \right\|_{\Gamma_h} \\
&\leqslant \frac{c}{\alpha_h \sqrt{\varepsilon}}
\end{aligned}$$

This gives:

$$\left\| \frac{d}{d\varepsilon} \alpha_h \tilde{F}_{z,h}(\varepsilon) \right\| \leqslant c \left(\varepsilon^{-\frac{1}{s}} \left\| \alpha_h \tilde{F}_{z,h}(\varepsilon) \right\|^{1-\frac{1}{s}} + \varepsilon^{-\frac{1}{2}} \right)$$

which, together with (5.9), gives the result. □

Let $\Omega_0 =]-1, 1[$ and $\Omega_j = \left\{ \lambda \in \mathbb{R} : 2^{j-1} \leqslant |\lambda| < 2^j \right\}$ for $j \in \mathbb{N}^*$. For a selfadjoint operator F on $\mathcal{H}$ and $s \geqslant 0$, the abstract Besov space $B_s(F)$ is defined by:

$$B_s(F) = \left\{ u \in \mathcal{H} : \|u\|_{B_s(F)} < \infty \right\}$$

where:

$$\|u\|_{B_s(F)} = \sum_{j \in \mathbb{N}} 2^{js} \left\| \mathbb{1}_{\Omega_j}(F) u \right\|_{\mathcal{H}}$$

The norm of its dual space $B_s^*(F)$ with respect to the scalar product on $\mathcal{H}$ is:

$$\|v\|_{B_s^*} = \sup_{j \in \mathbb{N}} 2^{-js} \left\| \mathbb{1}_{\Omega_j}(F) v \right\|_{\mathcal{H}}$$

When F is the multiplication by x on $L^2(\mathbb{R}^n)$ we recover the usual Besov spaces B_s and B_s^* and the norm we have just defined for B_s^* is equivalent to the usual one:

$$\sup_{R \geqslant 1} R^{-s} \left(\int_{|x| < R} |v(x)|^2 \, dx \right)^{\frac{1}{2}}$$

Theorem 5.3. *Let (H_h) be an abstract family of dissipative operators as in section 2, (A_h) a conjugate family for (H_h) on J with bounds (α_h) as in definition 2.1 and $s \geqslant \frac{1}{2}$. Then for all closed subinterval I of J there exists $c \geqslant 0$ such that for any $z \in \mathbb{C}_{I,h}$ and $h \in]0,1]$ we have:*

$$\left\|(H_h - z)^{-1}\right\|_{B_s(A_h) \to B_s^*(A_h)} \leqslant \frac{c}{\alpha_h}$$

Now that we have theorem 2.3 and proposition 5.2, we can follow word by word the proof of the analog theorem for selfadjoint operators (see theorem 2.2 in [Wan07]). Applied to our dissipative Schrödinger $H_h = -h^2\Delta + V_1(x) - i\nu(h)V_2(x)$, this gives:

Theorem 5.4. *Let $E > 0$ and $s \geqslant \frac{1}{2}$. If all bounded trajectories of energy E meet $\mathcal{O}$, then there exists $\varepsilon, h_0 > 0$ and $c \geqslant 0$ such that with $J = [E-\varepsilon, E+\varepsilon]$ we have for all $z \in \mathbb{C}_{J,+}$ and $h \in]0, h_0]$:*

$$\left\|(H_h - z)^{-1}\right\|_{B_s \to B_s^*} \leqslant \frac{c}{h\tilde{\nu}(h)}$$

(we recall that $\tilde{\nu}(h) = \min(1, \nu(h)/h)$).

Proof. We already have a conjugate family $(\tilde{\nu}(h)F_h)$ for (H_h). So we only have to apply the abstract theorem 5.3, (3.10), and the estimate:

$$\left\|(H_h - i)^{-1}\right\|_{B_s \to B_s(F_h)} \leqslant c \tag{5.10}$$

with a similar estimate for dual spaces. To prove (5.10), we use the idea given in [Hör84, 14.1]. For any $u \in B_s(F)$ and $k \in \mathbb{N}$, since the $\mathbb{1}_{\Omega_j}(F)u$ for $j \in \mathbb{N}$ are pairwise orthogonal we have:

$$\begin{aligned}
&\|u\|_{B_s(F_h)} \\
&\quad = \sum_{0 \leqslant j \leqslant k} 2^{js}\left\|\mathbb{1}_{\Omega_j}(F_h)u\right\| + \sum_{j>k} 2^{-js}\left\|2^{2js}\mathbb{1}_{\Omega_j}(F_h)u\right\| \\
&\quad \leqslant \left(\sum_{j \leqslant k} 2^{2js}\right)^{\frac{1}{2}}\left(\sum_{j \leqslant k}\left\|\mathbb{1}_{\Omega_j}(F_h)u\right\|^2\right)^{\frac{1}{2}} + \left(\sum_{j>k} 2^{-2js}\right)^{\frac{1}{2}}\left(\sum_{j>k}\left\|2^{2js}\mathbb{1}_{\Omega_j}(F_h)u\right\|^2\right)^{\frac{1}{2}} \\
&\quad \leqslant c_s 2^{ks}\|u\| + c_s 2^{-ks}\left\|\langle F_h\rangle^{2s} u\right\|
\end{aligned}$$

and hence, for $\varphi \in B_s$, using the fact that the operator $\langle F_h\rangle^{2s}(H_h - i)^{-1}\langle x\rangle^{-2s}$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ uniformly in h we have:

$$\begin{aligned}
\left\|(H_h - i)^{-1}\varphi\right\|_{B_s(F_h)} &\leqslant \sum_{k \in \mathbb{N}}\left\|(H_h - i)^{-1}\mathbb{1}_{\Omega_k}(x)\varphi\right\|_{B_s(F_h)} \\
&\leqslant c_s\sum_{k \in \mathbb{N}} 2^{ks}\left\|(H_h - i)^{-1}\mathbb{1}_{\Omega_k}(x)\varphi\right\| + c_s\sum_{k \in \mathbb{N}} 2^{-ks}\left\|\langle X\rangle^{2s}\mathbb{1}_{\Omega_k}(x)\varphi\right\| \\
&\leqslant c_s\sum_{k \in \mathbb{N}} 2^{ks}\left\|\mathbb{1}_{\Omega_k}(x)\varphi\right\| + c_s\sum_{k \in \mathbb{N}} 2^{-ks}\left\|\langle X\rangle^{2s}\mathbb{1}_{\Omega_k}(x)\varphi\right\| \\
&\leqslant c_s\|\varphi\|_{B_s} + c_s\sum_{k \in \mathbb{N}} 2^{ks}\left\|\mathbb{1}_{\Omega_k}(x)\varphi\right\| \\
&\leqslant c_s\|\varphi\|_{B_s}
\end{aligned}$$

□

Appendix A. Unitary dilations and dissipative Schrödinger operators

In order to use the selfadjoint theory to study dissipative operators, we have mostly used the assumption that H is a perturbation of its selfadjoint part H_1. However, by the theory of unitary dilations, there are other selfadjoint operators we can use:

Definition A.1. Let T be a bounded operator of the Hilbert space $\mathcal{H}$. A bounded operator U on a Hilbert space $\mathcal{K}$ is said to be a dilation of T if $\mathcal{H} \subset \mathcal{K}$ and for all $\varphi, \psi \in \mathcal{H}$ and $n \in \mathbb{N}$ we have:

$$\langle U^n \varphi, \psi \rangle_{\mathcal{K}} = \langle T^n \varphi, \psi \rangle_{\mathcal{H}}$$

The theory of unitary dilations for a contraction is developped in the book of B.S.-Nagy and C. Foias ([NF67]). In particular, we know that every contraction has a unitary dilation. This also holds for semigroups of contractions: if $(T(t))_{t \geqslant 0}$ is a semigroup of contractions on $\mathcal{H}$, there exists a unitary group $(U(t))_{t \in \mathbb{R}}$ on $\mathcal{K} \supset \mathcal{H}$ such that $U(t)$ is a dilation of $T(t)$ for all $t \geqslant 0$. Then, if H is a dissipative operator on $\mathcal{H}$, there is a unitary group of dilations $(U(t))$ on $\mathcal{K} \supset \mathcal{H}$ for the semigroup (e^{-itH}) generated by H. The unitary group $(U(t))$ is generated by a selfadjoint operator K on $\mathcal{K}$, the properties of which we can use to study the dissipative operator H. Note that K is usually said to be a selfadjoint dilation of H but is not a dilation of H in the sense of definition A.1.

Much is said on the abstract theory in [NF67], but there is an explicit study of the dissipative Schrödinger operator case in [Pav77]. In particular an exemple of dilation is given. Here we recall this example in the semiclassical setting:

Proposition A.2. *Let $-h^2\Delta + V_1 - ihV_2$ be a dissipative Schrödinger operator on $L^2(\mathbb{R}^n)$ as in section 3, $W_h = \sqrt{2hV_2}$, $\Omega = \operatorname{supp} V_2$, $\mathcal{K} = L^2(\mathbb{R}_-, L^2(\Omega)) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}_+, L^2(\Omega))$ and P the orthogonal projection of $\mathcal{K}$ on $L^2(\mathbb{R}^n)$. Then the operator:*

$$K_h : \varphi = (\varphi_-, \varphi_0, \varphi_+) \mapsto \left(-i\varphi_-', H_1^h \varphi_0 - \frac{W_h}{2}(\varphi_-(0) + \varphi_+(0)), -i\varphi_+' \right)$$

with domain:

$$\mathcal{D}(K_h) = \left\{ (\varphi_-, \varphi_0, \varphi_+) : \varphi_\pm \in H^1(\mathbb{R}_\pm, L^2(\Omega)) \text{ and } \varphi_+(0) - \varphi_-(0) = iW_h\varphi_0 \right\} \subset \mathcal{K}$$

(where H^1 is the Sobolev space of L^2-functions with first derivative in L^2) is a selfadjoint operator which satisfies:

$$\begin{aligned}
\forall z \in \mathbb{C}_+, \qquad & P(K_h - z)^{-1}\big|_{L^2(\mathbb{R}^n)} = (H_h - z)^{-1} \\
\forall z \in \mathbb{C}_+, \qquad & P(K_h - \overline{z})^{-1}\big|_{L^2(\mathbb{R}^n)} = (H_h^* - \overline{z})^{-1} \\
\forall t \geqslant 0, \qquad & P e^{-\frac{it}{h}K_h}\Big|_{L^2(\mathbb{R}^n)} = e^{-\frac{it}{h}H_h} \\
\forall t \leqslant 0, \qquad & P e^{-\frac{it}{h}K_h}\Big|_{L^2(\mathbb{R}^n)} = e^{-\frac{it}{h}H_h^*}
\end{aligned}$$

Proof. The proof of the proposition is straightforward calculations. We first have to check that K is symmetric, that $\mathcal{D}(K^*) \subset \mathcal{D}(K)$ and then that for $z \in \mathbb{C}_+$ we have $(\psi_-, \psi_0, \psi_+) = (K - z)^{-1}(\varphi_-, \varphi_0, \varphi_+)$ where:

$$\begin{aligned}
\psi_-(r) &= i\int_{-\infty}^{r} e^{iz(r-s)}\varphi_-(s)\, ds \\
\psi_0 &= (H_h - z)^{-1}(\varphi_0 + W_h\psi_-(0)) \\
\psi_+(r) &= (\psi_-(0) + iW_h\psi_0)e^{izr} + i\int_0^r e^{iz(r-s)}\varphi_+(s)\, ds
\end{aligned}$$

and an analog for $(K - \overline{z})^{-1}$. To prove the last statement, we show that the generator of the semigroup $t \mapsto Pe^{-\frac{it}{h}K_h}\Big|_{L^2(\mathbb{R}^n)}$ must be H using the result on the resolvent. Details are given in [Pav77]. ☐

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